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LETTER TO THE EDITOR

A new PT -symmetric complex Hamiltonian with a real spectrumB Bagchi[†] and R Roychoudhury[‡][†] Department of Applied Mathematics, University of Calcutta, 92 Acharya Prafulla Chandra Road, Calcutta 700009, India[‡] Physics and Applied Mathematics Unit, Indian Statistical Institute, 203 Barrackpore Trunk Road, Calcutta 700035, India

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Abstract. We construct an isospectrum system in terms of a real and a complex potential to show that the underlying PT -symmetric complex Hamiltonian possesses a real spectrum which is shared by its real partner.

Complex potentials have found wide usage in the literature (see [1] for an early work), especially in connection with scattering problems. Recently it has been emphasized [2] that by enforcing a PT symmetry, one can obtain new classes of complex Hamiltonian which exhibit a real spectrum of energy eigenvalues. The purpose of this letter is to bring to light a new complex Hamiltonian which is PT symmetric and possesses a real energy spectrum.

Consider potentials of the form $V^{(1),(2)} = U^2 \pm U'$, where U is complex function of x and a prime denotes a derivative with respect to x . Let us express U explicitly as $a(x) + ib(x)$, where $a(x)$ and $b(x)$ are certain real, continuously differentiable functions in R . We have

$$V^{(1),(2)} = (a^2 - b^2 \pm a') + i(2ab \pm b'). \quad (1)$$

In the following we investigate the case when one of the potentials defined by (1) is real but the other is complex. To this end we choose, for the sake of concreteness, $V^{(2)}$ to be real, thus restricting the function a to be given by

$$a = \frac{1}{2} \frac{b'}{b}. \quad (2)$$

The constraint (2) gives for $V^{(1)}$ and $V^{(2)}$:

$$V^{(1)} = (a^2 - b^2 + a') + 2ib' \quad (3a)$$

$$V^{(2)} = (a^2 - b^2 - a'). \quad (3b)$$

Notice that $V^{(1)} = V^{(2)} + 2U'$, which seems to hint at a supersymmetric connection between $V^{(1)}$ and $V^{(2)}$; however, it should be borne in mind that the Hamiltonians for $V^{(1)}$ and $V^{(2)}$ cannot be made simultaneously Hermitian.

Our task now is to demonstrate, by appropriately choosing the functions a and b , that the Hamiltonians $H^{(1),(2)} = -\frac{d^2}{dx^2} + V^{(1),(2)}$ yield a common real spectrum, thus forming an isospectral system. We propose the following example:

$$a = -\frac{\mu}{2} \tanh \mu x \quad b = \lambda \operatorname{sech} \mu x \quad (4)$$

Table 1. Eigenfunctions and eigenvalues of the potentials $V^{(1),(2)}$ for the case $\bar{\lambda} = 3$. See the text for details. N_0 and $N_1^{(i)}$ are normalization constants.

Eigenfunctions	For the potential $V^{(1)}$ ($i = 1$)	For the potential $V^{(2)}$ ($i = 2$)	Energy eigenvalues
$\psi_{(n=0)}^{(i)}$	$N_0 \operatorname{sech}^2 \mu x (\tanh \mu x + i \operatorname{sech} \mu x)$	$N_0 \operatorname{sech}^2 \mu x$	$\frac{\mu^2}{4} - 4\mu^2$
$\psi_{(n=1)}^{(i)}$	$N_1^{(1)} \operatorname{sech} \mu x (1 - \frac{5}{3} \operatorname{sech}^2 \mu x + i \frac{5}{3} \operatorname{sech}^2 \mu x \sinh \mu x)$	$N_1^{(2)} \operatorname{sech}^2 \mu x \sinh \mu x$	$\frac{\mu^2}{4} - \mu^2$

Under parity (P) $p \rightarrow -p$ and $x \rightarrow -x$, whereas under time reversal (T) $p \rightarrow -p$, $x \rightarrow -x$ and $i \rightarrow -i$.

where μ and λ are non-zero real parameters with $\mu \neq \lambda$. Note that the above representations of the functions a and b are consistent with requirement (2).

Substitution of (4) into (3) gives for $V^{(1)}$ and $V^{(2)}$ the expressions

$$V^{(1)} = \frac{\mu^2}{4} - \mu^2 [\bar{\lambda}(\bar{\lambda} - 1) + 1] \operatorname{sech}^2 \mu x - 2i\lambda\mu \operatorname{sech} \mu x \tanh \mu x \quad (5a)$$

$$V^{(2)} = \frac{\mu^2}{4} - \mu^2 \bar{\lambda}(\bar{\lambda} - 1) \operatorname{sech}^2 \mu x \quad (5b)$$

where $\bar{\lambda}(\bar{\lambda} - 1) = \frac{\lambda^2}{\mu^2} - \frac{1}{4}$.

It is well known [3] that the non-zero energy levels for $V^{(2)}$ are given by

$$E_n^{(2)} = \frac{\mu^2}{4} - (\bar{\lambda} - 1 - n)^2 \mu^2 \quad n < \bar{\lambda} - 1 \quad (6)$$

with $n = 0, 1, \dots$. The associated eigenfunctions for the even and odd states are

$$\psi_{\text{even}}^{(2)}(x) = \cosh^{\bar{\lambda}} \mu x {}_2F_1 \left[\frac{1}{2}(\bar{\lambda} - 1), \frac{1}{2}(\bar{\lambda} + 1), \frac{1}{2} - \sinh^2 \mu x \right] \quad (7)$$

$$\psi_{\text{odd}}^{(2)}(x) = \cosh^{\bar{\lambda}} \mu x \sinh \mu x {}_2F_1 \left(\frac{\bar{\lambda}}{2}, \frac{\bar{\lambda}}{2} + 1, \frac{3}{2}, -\sinh^2 \mu x \right). \quad (8)$$

The main point of this letter is to expose the fact that the complex Hamiltonian $H^{(1)}$, in addition to the zero-energy level, mimicks the discrete real spectrum (7) for a class of complex eigenfunctions. We illustrate this by considering the case $\frac{\lambda}{\mu} = -\frac{5}{2}$ for which $\bar{\lambda} = 3$. The relevant values of n then are 0 and 1, none of which, however, correspond to the zero-energy state for either $V^{(1)}$ or $V^{(2)}$. Our results, obtained by exploiting the intertwining relations $\psi_{(n=0,1)}^{(1)} = (\frac{d}{dx} + U)\psi_{(n=0,1)}^{(2)}$, are shown in table 1. It is obvious from the results that not only are the energy eigenvalues for $V^{(1)}$ real and match those of $V^{(2)}$ for $n = 0$ and 1 respectively, but also that the corresponding eigenfunctions for both $V^{(1)}$ and $V^{(2)}$ have controllable asymptotic behaviour.

Further, it is clear that the complex potential $V^{(1)}$ has a normalizable zero-energy state with the wavefunction

$$\psi_0^{(1)} \sim \sqrt{\operatorname{sech} \mu x} e^{2i \frac{\lambda}{\mu} \tan^{-1}(e^{\mu x})}. \quad (9)$$

This is indeed a nice result considering the fact that the imaginary part of $V^{(1)}$ only contributes a phase factor in $\psi_0^{(1)}$.

To conclude, we have found a new PT -symmetric complex potential whose energy levels are negative semi-definite and, excluding the zero-energy state, coincide with those of a known sech^2 potential. Results for different values of $\bar{\lambda}$ and other choices of a and b will be communicated in a future detailed publication.

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